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Resolving Singularities in a Ramsey-type Growth Model*

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Abstract. The aim of this paper is to discuss a mathematical solution procedure to solve a Ramsay-type growth model that explains the fundamentals of consumption and capital accumulation in a dynamic equilibrium setting. The problem is formulated as a system of recursive equations and studied through some numerical experiments for the time path of the different variables of the model under some alternative assumption for the steady-state equilibrium of the labour market conditioning the possible singularity of the model.

Key words: Equilibrium problems in macroeconomics; Ramsey-type growth model; Singular recursive systems

1. Introduction

The Ramsey growth model is a basic model in macroeconomics that explains the fundamentals of consumption and capital accumulation in a dynamic real equilibrium setting. It develops the standard Solow (1956) growth model by taking into account an endogeneous determination of the level of savings. This framework describes a closed economy populated by infinitely living firms and households. As in this economy markets are perfectly competitive, the allocation of resources achieved is Pareto optimal and is the same under a command and a decentralized equilibrium. This provides a useful benchmark for studying the optimal intertemporal allocation of resources under various types of imperfections.

The way the Ramsay model is solved by economists and mathematicians appears quite different. Economists put aside some mathematical particularities of this setting, in particular the fact that this model, once it is solved globally presents some singularities. These are due, for example, to the fact that the steady state around which economists treat the problem, assumes full employment. This feature creates problems for the mathematicians. On the other hand, mathematicians would adopt some solution procedure putting aside this singularity problem by assuming, for example, that steady-state labour supply and demand are different. This assumption is rejected in the economic analysis, as the Ramsey model belongs to a classical

* This paper is dedicated to the memory of Professor P.D. Panagiotopoulos.

approach of studying growth theory in a classical full employment setting. By such, a simplified global solution procedure in mathematics requires some particular assumptions in economics.

The aim of this paper is to bridge the gap between these two ways of solving the Ramsey growth model. To do so, we apply a global resolution method that allows one to treat the economic problem by taking into account explicitly both the labour and the goods market (the capital market is omitted according to Walras law). The version of the model that is adopted in this paper presents some differences from standard textbooks or research articles that adopt this framework (see, for example, Blanchard and Fischer (1989) or Barro and Sala-i-Martin (1995) for a standard presentation of the Ramsey model). It is presented in discrete time, and variables are not expressed in terms of level by inhabitants, in order to outline the influence of the assumption of full or underemployment in steady state, which in turn determines the possibility of singularities in the model. By such, the weight that is given to the labour market is quite heavier that is usually done.

The paper is organized as follows: the first part presents a deterministic discrete time version of this model. We study the determination of the equilibrium in the decentralized version of the model to provide a precise description of all the agents. In the second part we present the first-order conditions that characterize the optimal behaviour of the private sector and the log-linearized model of this version of the Ramsey model around its steady state, in order to outline its recursive structure. The third part presents the mathematical treatment that is used for solving the model, while the last part explores some numerical experiments for the time path of the different variables of the model under alternative assumption for the steady-state equilibrium of the labour markets conditioning the possible singularity of the model.

2. A decentralized macroeconomic setting

The model describes a decentralized closed economy, made up of three markets (for labour, capital and goods and services). As it is populated by identical households and firms, it is simpler to solve the model for a representative household and a representative firm. Both types of agents have perfect foresight and live for ever. At each point of time, the representative household decides how much labour and capital it wants to rent to firms and how much to save and consume in order to maximize its welfare. As it can choose to keep its savings either by accumulating capital or financial assets, the rental cost of capital must equal the net rate of interest. The representative firm rent the services of capital and labour to produce output. Both agents behave competitively, so they take as given both the real wage and the real rate of interest. Their decisions are made compatible at the macroeconomic level through the working of markets with flexible wages and real rate of interest. Applying Walras law allows one to solve the general equilibrium of the model only with reference to both the labour and the goods and services market.

This subsection describes more formally these ideas and outlines the main differences between the version of the model that is used in this paper and its standard economics textbook version.

2.1. THE REPRESENTATIVE HOUSEHOLD

In this economy, households provide labour services in exchange for wages, receive interest income on assets, purchase goods for consumption and save by accumulating additional assets. Assuming a rental cost of capital equal to the real rate of interest, allows one to simplify the model by assuming that households hold assets only in the form of ownership claims on capital. These decisions are made at each point of time in order to maximize a welfare function V depending upon consumption. In this paper we keep Ramsey's assumption concerning the fact that utility does not depend on work effort. assuming time separability in the preferences of this agent, allows one to define this function as the sum of the present value of current and future instant utility, i.e.:

$$V_{t_0} = \sum_{s=t_0}^{M} \beta^{s-t_0} U(C_{s-t_0})$$

where $M \in \mathbb{N}$, (M great), C_t is consumption at time t, $0 < \beta < 1$ is the psychological discount factor that is defined according to $\beta = 1/(1+\rho)$ (ρ representing the rate of time preference). Standard assumptions are made on $U(\cdot)$, so that it is a non-negative and concave increasing function of consumption. In this paper, we specialize $U(\cdot)$ to the Constant Intertemporal Elasticity of Substitution (CIES) case, so that the objective function of the representative agent that is used below can be written as:

$$V_{t_0} = \sum_{s=t_0}^{M} \beta^{s-t_0} \frac{C_{s-t_0}^{1-1/\sigma}}{1-1/\sigma} \tag{1}$$

where $\sigma \geqslant 1$ represents the elasticity of intertemporal substitution in the household consumption across periods. A smaller value indicates that the representative agent wishes to keep a relatively more stable consumption level from period to period. Inversely, if $\sigma \to \infty$ the agent does not care on its consumption profile, so that the level of C_s can highly fluctuate between two points of time. If $\sigma = 1$, utility is logarithmic. Setting $t := s - t_0$, $\gamma := 1 - 1/\sigma$ and $N := M - t_0$ (1) reduces to

$$V_0 = \sum_{t=0}^{N} \beta^t \frac{C_t^{\gamma}}{\gamma}\,. \tag{2}$$

The representative agent maximizes (1) given its Budget constraint, that is defined for period t according to

$$W_t L_t + R_t K_{t-1} = C_t + K_t\,. \tag{3}$$

It takes as given the gross real interest rate $R_t = (1 + r_t)$ where r_t is the real interest rate in the economy between $(t-1)$ and t as well as the real wage (W_t) that it receives for working L_t. In (3) K_{t-1} represents the stock of capital available at period $(t-1)$ by the representative household. The economic problem of the household in period t is to choose the value of (C_t, K_t) so that it can maximize (2) given (3). Labour effort is not a control variable in the consumer problem. As will be outlined below, we simplify this aspect by assuming that the agent passively adjusts its labour supply to labour demand in order to maintain the equilibrium of the labour market. Although this assumption would appear economically tricky it simplifies the mathematical computation of the problem without affecting the main results concerning the resolution of the singularity of the model.

2.2. THE REPRESENTATIVE FIRM

The representative firm produces output according to a macroeconomic production function that combines capital and labour. For convenience we assume that it has access to a standard Cobb–Douglas technology:

$$Y_t = A_t K_{t-1}^{\alpha} (L_t^d)^{1-\alpha} \tag{4}$$

where Y_t represents the level of goods supply, A_t the state of technology, L_t^d the quantity of labour it can hire on the market and $0 < \alpha < 1$. We assume that capital depreciates at a rate of $\delta \in]0, 1[$, per unit of time.

At each point of time, the objective of the representative firm is to maximize its profit defined here as the difference between output and the real cost of production. The latter is made up of two components: salaries $W_t L_t^d$ and the cost of using capital $(r_t + \delta)K_{t-1}$. The first term of this last component defines the rental cost of capital that is paid to the household, while δK_t represents the capital worn out cost. For convenience, we re-express the cost of using capital as $[(R_t - 1) + \delta]K_{t-1}$. Given these elements, the representative firm chooses the value of (L_t^d, K_t), in order to maximize its profit function defined according to

$$\Pi_t = A_t K_{t-1}^{\alpha} (L_t^d)^{1-\alpha} - W_t L_t^d - [(R_t - 1) + \delta]K_{t-1} \tag{5}$$

2.3. EQUILIBRIUM CONDITIONS

In this non-monetary economy, agents interact on the labour market, on the goods and services market and on the financial market. Applying Walras law, one just needs to characterize equilibrium on the first two:

$$L_t^s = L_t^d \tag{6}$$

and

$$I_t = S_t \tag{7}$$

where $I_t = K_t - K_{t-1}$ represents the level of investment in the economy and

$S_t = Y_t - C_t$ the level of savings. In this model, the labour market determines the wage level, while the market for goods and services determines the real interest rate.

3. A recursive presentation of the Ramsey problem

Having presented the decentralized economic problem, we can now go through the derivation of the first-order conditions that characterize the optimal behavior of the representative agents. Once obtained, these expressions are log-linearized around the steady state of the model in order to outline the recursive structure of the economic problem in this modified Ramsey framework. This treatment is implemented successively for both the representative agents taken individually and, eventually, for the economy as a whole.

3.1. THE REPRESENTATIVE HOUSEHOLD PROBLEM

Given the assumptions presented above, the representative household solves

$$\max_{\{C_t,K_t\}} \sum_{i=0}^{N} \beta^i \frac{C_i^\gamma}{\gamma} \tag{8}$$

subject to

$$W_t L_t^s + R_t K_{t-1} = C_t + K_t \, . \tag{9}$$

Forming the Lagrangian

$$L := \sum_{i=0}^{N} L_i$$

where

$$L_i := \beta^i \frac{C_i^\gamma}{\gamma} + \lambda_i (W_i L_i^s + R_i K_{i-1} - C_i - K_i)$$

and using the expression

$$L = \sum_{i=0}^{t-1} L_i + \beta^t \frac{C_t^\gamma}{\gamma} + \lambda_t (W_t L_t^s + R_t K_{t-1} - C_t - K_t) + \beta^{t+1} \frac{C_{t+1}^\gamma}{\gamma}$$
$$+ \lambda_{t+1} (W_{t+1} L_{t+1}^s + R_{t+1} K_t - C_{t+1} - K_{t+1}) + \sum_{i=t+2}^{N} L_i \, ,$$

we write the first-order optimality conditions

$$\frac{\partial L}{\partial \lambda_t} = 0, \frac{\partial L}{\partial C_t} = 0, \frac{\partial L}{\partial K_t} = 0 \, .$$

Here we have

$$\frac{\partial L}{\partial \lambda_t} = W_t L_t^s + R_t K_{t-1} - C_t - K_t \,, \tag{10}$$

$$\frac{\partial L}{\partial K_t} = -\lambda_t + R_{t+1} \lambda_{t+1} \,, \tag{11}$$

$$\frac{\partial L}{\partial C_t} = \beta^t C_t^{\gamma - 1} - \lambda_t \,. \tag{12}$$

Defining the index of relative consumption risk aversion as $\eta := 1 - \gamma = 1/\sigma$, we may write

$$\lambda_t = \beta^t C_t^{-\eta}$$

and use this last expression in (10) and (11) to obtain (for any $1 \leq t \leq N-1, t \in \mathbb{N}$) a system made of the budget constraint and of the intertemporal Euler condition:

$$W_t L_t^s + R_t K_{t-1} - C_t - K_t = 0 \,, \tag{13}$$

$$-C_t^{-\eta} + \beta R_{t+1} C_{t+1}^{-\eta} = 0 \,. \tag{14}$$

Formula (14) indicates that the household cannot gain from a shift in its consumption across time, once it has reached the maximum level of welfare. Indeed a unitary consumption reduction in period t lowers its utility by $-C_t^{-\eta}$. The consumption unit thus saved can be converted in capital that allows, in period $(t+1)$, R_{t+1} units of consumption. This, in turn, increases utility by $R_{t+1} \beta C_{t+1}^{-\eta}$. The intertemporal Euler condition thus simply states that at the optimum these two quantities are the same.

The objective function in (8) is concave while the constraint in (9) is linear. Moreover, the bordered Hessian associated to Problems (8) and (9) is given by

$$\overline{H}_t(K_t, L_t) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & \beta^t(\gamma - 1) C_t^{\gamma - 2} \end{pmatrix} .$$

It is clear that $\det \overline{H}_t > 0$ for $(K_t, C_t) \in \mathbb{R}_+ \times \mathbb{R}_+ \backslash \{0, 0\}$ so that the second-order optimality condition is satisfied on $\mathbb{R}_+ \times \mathbb{R}_+ \backslash \{0, 0)\}$.

Let us now apply the log-linearization principle (see, e.g., Uhlig (1998)) to the system (13)–(14). It consists of using a first-order Taylor approximation around the steady state so as to replace the nonlinear model by a linear one in the log-deviations of the variables. Formally, if X_t denotes a variable, then we may write

$$X_t = \overline{X} e^{x_t} \tag{15}$$

where $\overline{X}$ denotes the steady state and x_t the log-deviation of the variable X_t. Note that

$$x_t + \ln(\overline{X}) = \ln(X_t) \,. \tag{16}$$

On the other hand, for x_t small around zero, we have

$$e^{x_t} \cong 1 + x_t \,. \tag{17}$$

One first notes that the steady-states variables $\overline{K}$, $\overline{L}$, $\overline{W}$, $\overline{C}$, $\overline{R}$, satisfy the relations

$$\overline{WL}^s + \overline{RK} - \overline{C} - \overline{K} = 0 \,, \tag{18}$$

$$-\overline{C}^{-\eta} + \beta\overline{RC}^{-\eta} = 0 \,. \tag{19}$$

The constant $\overline{C}$ being nonzero, (19) is equivalent to $\beta\overline{R} - 1 = 0$, so that

$$\bar{r} = \rho$$

where $\bar{r} = \overline{R} - 1$, i.e., at the steady-state individual and social rate of time preference are the same. From (13) and (18), we obtain

$$\overline{K}k_t - \overline{RK}k_{t-1} + \overline{C}c_t = \overline{WL}^s w_t + \overline{WL}^s l_t^s + \overline{RK}r_t \,. \tag{20}$$

On the other hand, using (14) together with (19), we get

$$\beta\overline{R}\eta c_{t+1} - \eta c_t = \beta\overline{R}r_{t+1} \,.$$

The last relation being true for any $1 \leqslant t \leqslant N-1$, $t \in \mathbb{N}$, we may write

$$\beta\overline{R}\eta c_t - \eta c_{t-1} = \beta\overline{R}r_t \,.$$

for any $2 \leqslant t \leqslant N$, $t \in \mathbb{N}$. Assuming that the last relation holds also for $t = 1$ and using the relation $\beta\overline{R} = 1$, we obtain the recursive equation (for $1 \leqslant t \leqslant N$, $t \in \mathbb{N}$).

$$\eta c_t - \eta c_{t-1} = r_t \,. \tag{21}$$

Thus, the behaviour of the representative household can be characterized through the following recursive model:

$$A_1 X_t = B_1 X_{t-1} + F_t \,, \tag{22}$$

where

$$X_t = \begin{pmatrix} c_t \\ k_t \end{pmatrix}, \quad F_t = \begin{pmatrix} r_t \\ \overline{WL}^s w_t + \overline{WL}^s l_t^s + \overline{RK} r_t \end{pmatrix},$$

and

$$A_1 = \begin{pmatrix} \eta & 0 \\ \overline{C} & \overline{K} \end{pmatrix}, \quad B_1 = \begin{pmatrix} \eta & 0 \\ 0 & \overline{RK} \end{pmatrix}.$$

3.2. THE REPRESENTATIVE FIRM PROBLEM

Given the assumptions of the model, the representative firm solves (at time $t \geqslant 1$) the problem:

$$\max_{K_{t-1}, L_t^d} \Pi_t \,,$$

where Π_t is defined in (5).

The first-order conditions of optimality are

$$\frac{\partial \Pi_t}{\partial K_{t-1}} = 0\,, \quad \frac{\partial \Pi_t}{\partial L_t^d} = 0\,,$$

i.e., the return on production factor is given by their marginal productivity:

$$R_t = \alpha A_t K_{t-1}^{\alpha-1} (L_t^d)^{1-\alpha} + (1-\delta)\,, \tag{23}$$

and

$$W_t = (1-\alpha) A_t K_{t-1}^{\alpha} (L_t^d)^{-\alpha}\,. \tag{24}$$

Using (4), we can also write

$$R_t K_{t-1} = \alpha Y_t + (1-\delta) K_{t-1}\,, \tag{25}$$

and

$$W_t L_t^d = (1-\alpha) Y_t\,. \tag{26}$$

Relation (25) expresses that the returns equal the capital share plus one minus depreciation while relation (26) means that the wage payments equal the labour.

Note here that the Hessian matrix of Π_t is given by

$$H_t(K_{t-1}, L_t) = \begin{pmatrix} \alpha(\alpha-1) A_t K_{t-1}^{\alpha-2} (L_t^d)^{1-\alpha} & \alpha(1-\alpha) A_t K_{t-1}^{\alpha-1} (L_t^d)^{-\alpha} \\ \alpha(1-\alpha) A_t K_{t-1}^{\alpha-1} (L_t^d)^{-\alpha} & \alpha(\alpha-1) A_t K_{t-1}^{\alpha} (L_t^d)^{-\alpha-1} \end{pmatrix}.$$

It is easy to see that H_t is negative semidefinite as soon as $(K_{t-1}, L_t^d) \in \mathbb{R}_+ \times \mathbb{R}_+ \backslash \{9, 0)\}$. The second-order condition of optimality is therefore satisfied on $\mathbb{R}_+ \times \mathbb{R}_+ \backslash \{9, 0)\}$

Proceeding as before, let us first use the recursive model (4), (25) and (26) to deduce equalities satisfied by the steady-state variables $\overline{K}$, $\overline{L}^d$, $\overline{R}$ and $\overline{W}$, i.e.

$$\overline{WL}^d = (1-\alpha)\overline{Y} \tag{27}$$

$$\overline{Y} = \overline{AK}^{\alpha} (\overline{L}^d)^{1-\alpha} \tag{28}$$

$$\overline{RK} = \alpha\overline{Y} + (1-\delta)\overline{K}\,. \tag{29}$$

Let us now write the linearized model to describe the form program. From (4), we get

$$\ln(Y_t) = \ln(A_t) + \alpha \ln(K_{t-1}) + (1-\alpha)\ln(L_t^d) \tag{30}$$

and then using the 'log-deviation' notations as above and (30), we see that

$$y_t - \alpha k_{t-1} - (1-\alpha) l_t^d = a_t\,. \tag{31}$$

From (25), we get

$$\overline{RK}\, e^{r_t + k_{t-1}} = \alpha\overline{Y}\, e^{y_t} + (1-\delta)\overline{K}\, e^{k_{t-1}}$$

and the linearized model

$$\overline{RK}(1 + r_t + k_{t-1}) = \alpha\overline{Y}(1 + y_t) + (1 - \delta)\overline{K}(1 + k_{t-1}),$$

that is

$$\overline{K}(\overline{R} - (1 - \delta))k_{t-1} - \alpha\overline{Y}y_t = -\overline{RK} + \alpha\overline{Y} + (1 - \delta)\overline{K} - \overline{RK}r_t .$$

Then using (29), we may write

$$\alpha\overline{Y}k_{t-1} - \alpha\overline{Y}y_t = -\overline{RK}r_t . \tag{32}$$

Now, using (26), we obtain

$$\ln(W_t) + \ln(L_t^d) = \ln(1 - \alpha) + \ln(Y_t),$$

and thus using (27), we get

$$l_t^d - y_t = -w_t . \tag{33}$$

So, from Eqs. (31)–(33), we get the recursive model

$$A_2 Q_t = B_2 Q_{t-1} + G_t ,$$

where

$$Q_t = \begin{pmatrix} y_t \\ k_t \\ l_t^d \end{pmatrix}, \quad G_t = \begin{pmatrix} a_t \\ -\overline{RK}r_t \\ -w_t \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 1 & 0 & \alpha - 1 \\ -\alpha\overline{Y} & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

and

$$B_2 = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & -\alpha\overline{Y} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note here that the matrices A_2 and B_2 are singular.

3.3. THE EQUILIBRIUM PROBLEM

Combining the recursive relations obtained above, we can formulate the firm–agent equilibriun problem through the equilibrium conditions (6) and (7). So, the equilibrium problem is characterized by the following system of recursive equations

$$\eta c_t - r_t - \eta c_{t-1} = 0,$$
$$\overline{K}k_t + \overline{C}c_t - \overline{RK}k_{t-1} - \overline{WL}^s w_t - \overline{WL}^s l_t^s - \overline{RK}r_t = 0,$$
$$y_t - \alpha k_{t-1} - (1 - \alpha)l_t^d = a_t ,$$
$$-\alpha\overline{Y}y_t + \alpha\overline{Y}k_{t-1} + \overline{RK}r_t = 0,$$
$$l_t^d - y_t + w_t = 0,$$

$$l_t^s - l_t^d = \ln\left(\frac{\overline{L}^d}{\overline{L}^s}\right),$$
$$\overline{K}k_t - \overline{K}k_{t-1} - \overline{Y}y_t + \overline{C}c_t = \overline{Y} - \overline{C}$$

As noted above, the representative household passively adjusts the growth rate of its labour supply so as to compensate the difference between the growth rate of labour demand and the steady-state differential between labour supply and demand (sixth line of this system). This characteristic will induce the possibility of singularity in this version of the Ramsey model. In matrix form, we obtain the second-order recursive system:

$$AZ_t = BZ_{t-1} + H_t, \tag{34}$$

where

$$Z_t = \begin{bmatrix} c_t \\ r_t \\ k_t \\ y_t \\ w_t \\ l_t^s \\ l_t^d \end{bmatrix}, H_t = \begin{bmatrix} 0 \\ 0 \\ a_t \\ 0 \\ 0 \\ \ln\left(\frac{\overline{L}^d}{\overline{L}^s}\right) \\ \overline{Y} - \overline{C} \end{bmatrix},$$

$$A = \begin{bmatrix} \eta & -1 & 0 & 0 & 0 & 0 & 0 \\ \overline{C} & -\overline{RK} & \overline{K} & 0 & -\overline{WL}^s & -\overline{WL}^s & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \alpha - 1 \\ 0 & \overline{RK} & 0 & -\alpha\overline{Y} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ \overline{C} & 0 & \overline{K} & -\overline{Y} & 0 & 0 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} \eta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \overline{RK} & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha\overline{Y} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \overline{K} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

4. Mathematical treatment of the model

The aim of this section is to discuss the essential mathematical techniques we will use to solve the aforementioned model. The problems considered in this paper are formulated as linear systems of difference equations like

$$EU_t = FU_{t-1} + S_t$$

An inherent property which makes this system simple to deal is the superposition principle, ensuring that any solution can be written as the sum of the general solution of the associated homogeneous problem and a particular solution. A particular solution can be found by using the z-transform approach (see, e.g., Vich (1987)). If the matrix E is regular the general solution of the homogeneous problem can be found easily. If the matrix E is singular, then some appropriate methodology must be applied.

Let us first consider the homogeneous recursive system

$$EU_t = FU_{t-1} . \tag{35}$$

If the matrix E is regular then

$$U_t = E^{-1}FU_{t-1}$$

and the general solution $U_{h,t}$ of (35) is given by

$$U_{h,t} = (E^{-1}F)^t C$$

for some arbitrary vector C.

If the matrix E is singular then some appropriate mathematical methodology needs to be applied. Let us first consider the homogeneous problem. We suppose the existence of $\lambda \in \mathbb{R}$ such that $(\lambda E + F)$ is regular. One sets

$$\hat{E}_\lambda = (\lambda E + F)^{-1}E$$

and

$$\hat{F}_\lambda = (\lambda E + F)^{-1}F .$$

Remarking that $\lambda\hat{E}_\lambda + \hat{F}_\lambda = I$, we obtain in multiplying (35) by $(\lambda E + F)^{-1}$

$$\hat{E}_\lambda U_t = \hat{F}_\lambda U_{t-1} = (I - \lambda\hat{E}_\lambda)U_{t-1} .$$

The Jordan form of the singular matrix $\hat{E}_\lambda$ is

$$\hat{E}_\lambda = T\begin{pmatrix} W & 0 \\ 0 & N \end{pmatrix}T^{-1} ,$$

where W contains all the Jordan blocks corresponding to the nonzero eigenvalues of $\hat{E}_\lambda$ and N is nilpotent of order ν. We have

$$\begin{aligned}(I - \lambda\hat{E}_\lambda) &= TT^{-1} - \lambda T\begin{pmatrix} W & 0 \\ 0 & N \end{pmatrix}T^{-1} \\ &= T\begin{pmatrix} I - \lambda W & 0 \\ 0 & I - \lambda N \end{pmatrix}T^{-1} .\end{aligned}$$

So, from (35), we deduce the system

$$T\begin{pmatrix} W & 0 \\ 0 & N \end{pmatrix} T^{-1} U_t = T\begin{pmatrix} I-\lambda W & 0 \\ 0 & I-\lambda N \end{pmatrix} T^{-1} U_{t-1} . \tag{36}$$

Multiplying (36) by T^{-1} and setting $V_t^{(1)} = (T^{-1}U_t)^{(1)}$,

$$WV_t^{(1)} = (I-\lambda W)V_{t-1}^{(1)} , \tag{37}$$

$$NV_t^{(2)} = (I-\lambda N)V_{t-1}^{(2)} . \tag{38}$$

Relation (37) gives

$$V_t^{(1)} = W^{-1}(I-\lambda W)V_{t-1}^{(1)}$$

whose general solution is

$$V_t^{(1)} = (W^{-1}(I-\lambda W))^t D ,$$

where D is some arbitrary vector (with the appropriate size). From (38), we obtain that

$$V_{t-1}^{(2)} = (I-\lambda N)^{-1} N V_t^{(2)} = (I-\lambda N)^{-\nu} N^{\nu} V_{t+\nu-1}^{(2)} = 0 .$$

It results that necessarily $V_t^{(2)} = 0$. So, the general solution $U_{h,t}$ of (35) may be written as follows

$$U_{h,t} = T\begin{pmatrix} (W^{-1}(I-\lambda W))^t D \\ 0 \end{pmatrix} .$$

Let us now consider the nonhomogeneous system

$$EU_t = FU_{t-1} + S_t . \tag{39}$$

A particular solution $U_{p,t}$ may be found by means of the z-transform method. Indeed, let $U(z)$ and $S(z)$ denote the z-transforms of U_t and S_t, respectively. We have

$$EU(z) = z^{-1}FU(z) + S(z)$$

and thus

$$U(z) = (zE - F)^{-1} zS(z) . \tag{40}$$

If Γ is a closed Jordan curve around zero enclosing all the poles of the function $\varphi : \mathbb{C} \to \mathbb{C}$ defined by

$$\varphi(z) = (zE - F)^{-1} S(z) z^t$$

then

$$U_{p,t} = \frac{1}{2\pi i} \oint_{\Gamma} \varphi(z)\, \mathrm{d}z . \tag{41}$$

So, the general solution of Problem (39) is given by

$$U_t = U_{h,t} + U_{p,t} ,$$

that is

$$U_t = (E^{-1}F)^t C + \frac{1}{2\pi i}\oint_\Gamma (zE - F)^{-1} S(z) z^t \,\mathrm{dz}$$

if E is regular, and

$$U_t = T\begin{pmatrix} (W^{-1}(I - \lambda W))^t D \\ 0 \end{pmatrix} + \frac{1}{2\pi i}\oint_\Gamma (zE - F)^{-1} S(z) z^t \,\mathrm{dz}$$

if E is singular. In this last case we have

$$U_0 = T\begin{pmatrix} D \\ 0 \end{pmatrix} + \frac{1}{2\pi i}\oint_\Gamma (zE - F)^{-1} S(z) \,\mathrm{dz}\,,$$

so that the initial data must satisfy the compatibility conditions

$$U_0^{(2)} = \left(T\begin{pmatrix} D \\ 0 \end{pmatrix}\right)^{(2)} + \frac{1}{2\pi i}\oint_\Gamma ((zE - F)^{-1} S(z)^{(2)} \,\mathrm{dz}\,.$$

5. Numerical experiments

Let us now apply the mathematical treatment outlined in the previous paragraph to the system (34). Let us first remark that

$$\det(A) = \eta \overline{RK}^2 (1 - \alpha)(\overline{WL}^s - (1 - \alpha)\overline{Y})\,.$$

If $\det(A) \neq 0$ then problem (34) can be solved for any initial conditions. However, there exist various possibilities for A to be singular. Among them, one is particularly appealing in macroeconomics as it corresponds to full employment in the steady state:

$$\overline{L}^s = \overline{L}^d \tag{42}$$

In this case,

$$\det(A) = \eta \overline{RK}^2 (1 - \alpha)(\overline{WL}^d - (1 - \alpha)\overline{Y})\,,$$

and recalling (27), we see that det $(A) = 0$. It results that the matrix A is always singular in this case. In what follows we present the result of our numerical experiments for two situations: the first one in which condition (42) is violated, the second in which (42) holds.

5.1. THE REGULAR CASE

Suppose that $\sigma = 2$, $\delta = 0.1$, $\alpha = 0.7$, $\beta = 0.9615485$, $\overline{W} = 1$, $\overline{L}^s = 1$ and $\overline{L}^d = 1.5$. We deduce from these data and relations (18), (19), (27), (28) and (29) that $\eta = 0.5$, $\overline{A} = 0.465141$, $\overline{R} = 1.04$, $\overline{Y} = 5$, $\overline{K} = 25$ and $\overline{C} = 2$. Here $\det(A) \neq 0$ and the general form of the homogeneous solution is

$$U_{h,t} = (A^{-1}B)^t C$$

where C is some arbitrary vector in $\mathbb{R}^7$. Suppose now that $a_t = 0$ for $t < 0$ and $a_t = at + b$ for $t \geqslant 0$, we obtain

$$H(z) = \left(0 \quad 0 \quad \frac{az}{(z-1)^2} + \frac{bz}{z-1} \quad 0 \quad 0 \quad \frac{\ln\left(\frac{3}{2}\right)z}{z-1} \; \frac{3z}{z-1} \right)^T$$

and a particular solution is given as

$$H_t = \begin{bmatrix} \frac{5}{6}\ln\left(\frac{3}{2}\right) - 1 \\ 0 \\ \frac{1}{3}\ln\left(\frac{3}{2}\right) - 1 \\ \frac{1}{3}\ln\left(\frac{3}{2}\right) - 1 \\ \frac{10}{3}b + \frac{10}{3}at \\ \frac{4}{3}\ln\left(\frac{3}{2}\right) - 1 - \frac{10}{3}at - \frac{10}{3}b \\ \frac{1}{3}\ln\left(\frac{3}{2}\right) - 1 - \frac{10}{3}at - \frac{10}{3}b \end{bmatrix}.$$

Suppose now that the initial conditions are given by

$$U_0 = \begin{bmatrix} c_0 \\ r_0 \\ k_0 \\ y_0 \\ w_0 \\ l_0^s \\ l_0^d \end{bmatrix}.$$

Then we obtain

$$C = \begin{bmatrix} c_0 + 1 - \frac{5}{6}\ln\left(\frac{2}{3}\right) \\ r_0 \\ k_0 + 1 - \frac{1}{3}\ln\left(\frac{3}{2}\right) \\ y_0 + 1 - \frac{1}{3}\ln\left(\frac{3}{2}\right) \\ w_0 - \frac{10}{3}b \\ l_0^s + 1 - \frac{4}{3}\ln\left(\frac{3}{2}\right) + \frac{10}{3}b \\ l_0^d + 1 - \frac{1}{3}\ln\left(\frac{3}{2}\right) + \frac{10}{3}b \end{bmatrix}.$$

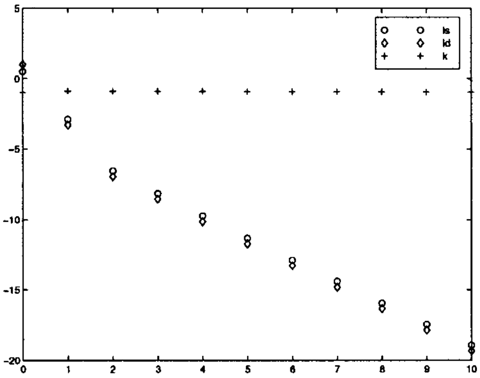

Figure 1.

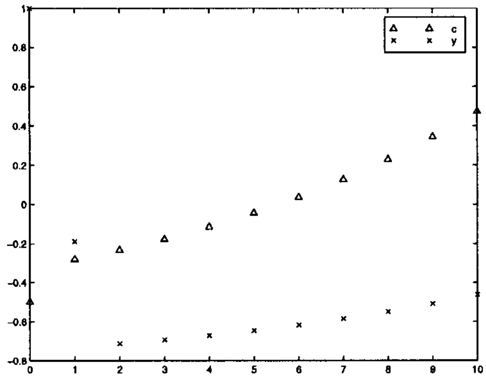

Figure 2.

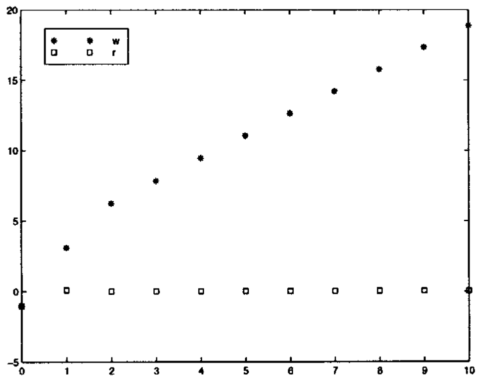

Figure 3.

The result that we obtain from the simulations are presented in Figures 1–3. This case is very particular in economics, as it assumes over-employment in the steady state (i.e., $\overline{L}^d > \overline{L}^s$). For the labour market to be in equilibrium (condition (6)), this requires either that $l_t^s > l_t^d > 0$ or $l_t^d < l_t^s < 0$. In both cases, the value of l_t^d determines the evolution of labour market equilibrium. This is graphically characterized by an acceleration in the rate of its decrease, as through (31), l_t^d is negatively related to $a_t > 0$. By the same relation, the positive rate of technological progress reduces the capital stock. Through Figure 1, it must be noted that, as the rate of reduction in the labour force is greater than that of capital desaccumulation, marginal productivity of labour rises which, in turn, increases the growth rate of the real wage (Figure 3). As both labour and capital growth rate are negative so is that of activity. In this situation, the adjustment of the goods market portrayed in Figure 2 can be explained as follows: (1) because of capital desaccumulation, investment falls. On the other hand, (2) as consumption growth rate is either slightly negative or is positive, while the rate of fall in output is always greater, savings decrease in this economy. The magnitude of these two effects is comparable, which eventually has no clear impact on the real interest rate to keep the equilibrium of the goods market.

5.2. THE SINGULAR CASE

The singular case corresponds to a situation in which full employment holds in the steady state. This situation is the one that is commonly postulated in economics. As shown in Figures 4–6, the time path of the different growth (or decrease) rates is totally different from the previous situation. The time paths are presented for the following value: $\sigma = 2$, $\delta = 0.1$, $\alpha = 0.7$, $\beta = 0.9615385$, $\overline{W} = 1$, $\overline{L}^s = 1$ and $\overline{L}^d = 1$. As in the previous case, we obtain that $\eta = 0.5$, $\overline{R} = 1.04$, $\overline{A} = 0.456141$, $\overline{Y} = 10/3$, $\overline{K} = 50/3$ and $\overline{C} = 5/3$. Here $\det(A) = 0$. On the other hand

$$\det(A + B) \neq 0$$

and

$$(A + B)^{-1} A = T \begin{pmatrix} w & 0 \\ 0 & N \end{pmatrix} T^{-1},$$

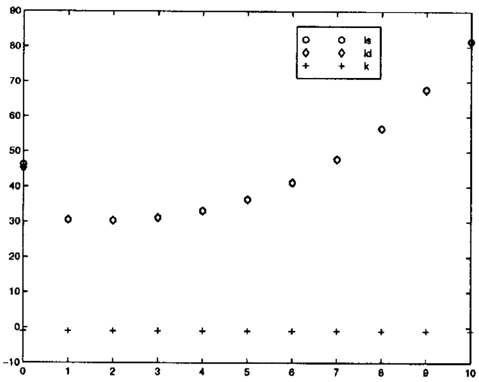

Figure 4.

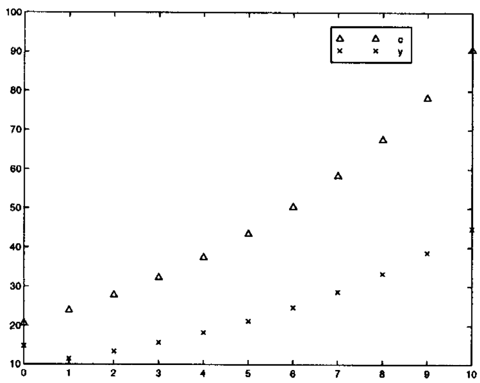

Figure 5.

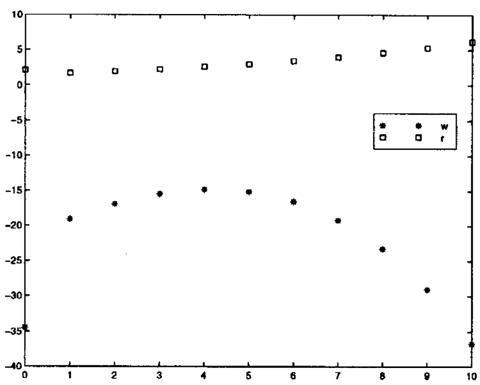

Figure 6.

with

$$W = \operatorname{diag}\left\{\frac{45}{97}, 1, 1, 1, 1, 1\right\},$$
$$N = 0$$

and

$$\begin{bmatrix} -\frac{194}{9} & 0 & 0 & 0 & 0 & 0 & \frac{14}{9} \\ -\frac{679}{468} & \frac{35}{52} & 0 & 0 & 0 & 0 & \frac{7}{9} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{97}{9} & 6 & 1 & 0 & 0 & 0 & \frac{52}{9} \\ \frac{679}{27} & -\frac{32}{3} & 1 & 1 & 0 & 0 & -\frac{364}{27} \\ -\frac{970}{27} & \frac{53}{3} & 1 & 1 & 1 & 0 & \frac{520}{27} \\ -\frac{970}{27} & \frac{53}{3} & 1 & 1 & 1 & 1 & \frac{520}{27} \end{bmatrix}.$$

So, the general form of the homogeneous solution is

$$U_{h,t} = T\begin{pmatrix} (W^{-1}(I-W))^t D \\ 0 \end{pmatrix},$$

where D is some arbitrary vector in $\mathbb{R}^6$. We have, for $t \geqslant 1$,

$$(W^{-1}(I-W))^t = \operatorname{diag}\left\{\left(\frac{97}{45}\right)^t, 0, 0, 0, 0, 0\right\}$$

and thus

$$(W^{-1}(I-W))^t D = \begin{pmatrix} \left(\frac{97}{45}\right)^t D_1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$$

It results that for $t \geqslant 1$,

$$U_{h,t} = \begin{bmatrix} -\frac{194}{9}\left(\frac{94}{45}\right)^t D_1 \\ -\frac{679}{468}\left(\frac{94}{45}\right)^t D_1 \\ 0 \\ -\frac{97}{9}\left(\frac{94}{45}\right)^t D_1 \\ \frac{679}{27}\left(\frac{94}{45}\right)^t D_1 \\ -\frac{970}{27}\left(\frac{94}{45}\right)^t D_1 \\ -\frac{970}{27}\left(\frac{94}{45}\right)^t D_1 \end{bmatrix}.$$

Let a_t be defined as in Example 5.1. Here we have

$$H(z) = \begin{pmatrix} 0 & 0 & \frac{az}{(z-1)^2} + \frac{bz}{z-1} & 0 & 0 & 0 & \frac{\frac{5}{3}z}{z-1} \end{pmatrix}^T$$

and a particular solution is given as

$$U_{p,t} = \begin{bmatrix} -1 \\ 0 \\ -1 \\ -1 \\ \frac{10}{3}ta + \frac{10}{3}b \\ -\frac{10}{3}ta - \frac{10}{3}b - 1 \\ -\frac{10}{3}ta - \frac{10}{3}b - 1 \end{bmatrix}.$$

Note that

$$U_{h,0} + U_{p,0} = T\left(\frac{D}{0}\right) + \begin{bmatrix} -1 \\ 0 \\ -1 \\ -1 \\ \frac{10}{3}b \\ -\frac{10}{3}b - 1 \\ -\frac{10}{3}b - 1 \end{bmatrix}.$$

The result that we obtain from the simulations are presented in Figures 4–6. The main difference between this situation and the regular case, comes from the fact that the equilibrium of the labour market at each point in time no longer requires a difference between the growth rates of labour supply and demand, and by so, is no longer determined primarily by relation (31) and $l_t^s = l_t^d$. In this case, for a positive realisation of a_t, the labour market displays an increase in employment and a reduction in the real wage, while the magnitude of the decrease in the capital stock is comparable as in the regular case. It is by such easy to explain the decrease in the real wage as the increase in the labour/capital ratio reduces the marginal productivity of labour. The positive growth rate of labour and the slightly negative value for capital lead to an increase in the growth rate of activity. On the goods market, because consumption preferences are biased towards the present, the growth rate of consumption is greater than that of activity and savings decrease more and more rapidly as time passes. As the growth rate of capital desaccumulation is constant, investment monotonically decreases, inducing an increase in the real interest rate to preserve the equilibrium of the goods market.

6. Conclusion

The aim of this paper has been to study the regularity of the Ramsey growth model. The way the problem has been presented allowed us to link the singularity of this model to the steady-state difference between labour supply and labour demand.

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